NOTE ON MMAT 5010: LINEAR ANALYSIS (2017 1ST TERM)

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1. Lecture 1: Normed spaces

Throughout this note, we always denote $\mathbb{K}$ by the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathbb{N}$ be the set of all natural numbers. Also, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$.

Definition 1.1. *Let X be a vector space over the field $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a normed space.
Also, the distance between the elements x and y in X is defined by $\|x - y\|$.

The following examples are important classes in the study of functional analysis.

Example 1.2. *Consider $X = \mathbb{K}^n$. Put*

$$\|x\|_p := \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p} \quad \text{and} \quad \|x\|_\infty := \max_{i=1,\dots,n} |x_i|$$

for $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$.
Then $\|\cdot\|_p$ (called the usual norm as p=2) and $\|\cdot\|_\infty$ (called the sup*-norm) all are norms on $\mathbb{K}^n$.*

Example 1.3. *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K},\ \lim |x(i)| = 0\} \text{(called the null sequnce space)}$$

and

$$\ell^\infty := \{(x(i)) : x(i) \in \mathbb{K},\ \sup_i |x(i)| < \infty\}.$$

Then c_0 is a subspace of ℓ^∞. The sup*-norm $\|\cdot\|_\infty$ on ℓ^∞ is defined by*

$$\|x\|_\infty := \sup_i |x(i)|$$

for $x \in \ell^\infty$. Let

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\}.$$

Also, c_{00} is endowed with the sup*-norm defined above and is called the finite sequence space.*

Example 1.4. *For $1 \leq p < \infty$, put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K},\ \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also, ℓ^p is equipped with the norm

$$\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$$

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for $x \in \ell^p$. Then $\|\cdot\|_p$ is a norm on ℓ^p (see [1, Section 9.1]*).*

Example 1.5. *Let $C^b(\mathbb{R})$ be the space of all bounded continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$. Now $C^b(\mathbb{R})$ is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$$

for every $f \in C^b(\mathbb{R})$. Then $\|\cdot\|_\infty$ is a norm on $C^b(\mathbb{R})$.
Also, we consider the following subspaces of $C^b(X)$.
Let $C_0(\mathbb{R})$ $\big($resp. $C_c(\mathbb{R})\big)$ be the space of all continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$ which vanish at infinity (resp. have compact supports), that is, for every $\varepsilon > 0$, there is a $K > 0$ such that $|f(x)| < \varepsilon$ (resp. $f(x) \equiv 0$) for all $|x| > K$.
It is clear that we have $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C^b(\mathbb{R})$.
Now $C_0(\mathbb{R})$ and $C_c(\mathbb{R})$ are endowed with the sup*-norm $\|\cdot\|_\infty$.*

Notation 1.6. *From now on, $(X, \|\cdot\|)$ always denotes a normed space over a field $\mathbb{K}$.*
For $r > 0$ and $x \in X$, let

- *(i) $B(x, r) := \{y \in X : \|x - y\| < r\}$ (called an open ball with the center at x of radius r) and $B^*(x, r) := \{y \in X : 0 < \|x - y\| < r\}$*
- *(ii) $B(x, r) := \{y \in X : \|x - y\| \le r\}$ (called a closed ball with the center at x of radius r).*

Put $B_X := \{x \in X : \|x\| \le 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ the closed unit ball and the unit sphere of X respectively.

Definition 1.7. *Let A be a subset of X.*

- *(i) A point $a \in A$ is called an interior point of A if there is $r > 0$ such that $B(a, r) \subseteq A$. Write $int(A)$ for the set of all interior points of A.*
- *(ii) A is called an open subset of X if $int(A) = A$.*

Example 1.8. *We keep the notation as above.*

- *(i) Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of all integers and rational numbers respectively If $\mathbb{Z}$ and $\mathbb{Q}$ both are viewed as the subsets of $\mathbb{R}$, then $int(\mathbb{Z})$ and $int(\mathbb{Q})$ both are empty.*
- *(ii) The open interval $(0, 1)$ is an open subset of $\mathbb{R}$ but it is not an open subset of $\mathbb{R}^2$. In fact, $int(0, 1) = (0, 1)$ if $(0, 1)$ is considered as a subset of $\mathbb{R}$ but $int(0, 1) = \emptyset$ while $(0, 1)$ is viewed as a subset of $\mathbb{R}^2$.*
- *(iii) Every open ball is an open subset of X (***Check!!***).*

Definition 1.9. *We say that a sequence (x_n) in X converges to an element $a \in X$ if $\lim \|x_n - a\| = 0$, that is, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - a\| < \varepsilon$ for all $n \ge N$.*
In this case, (x_n) is said to be convergent and a is called a limit of the sequence (x_n).

Remark 1.10.

(i) *If (x_n) is a convergence sequence in X, then its limit is unique. In fact, if a and b both are the limits of (x_n), then we have $\|a - b\| \le \|a - x_n\| + \|x_n - b\| \to 0$. So, $\|a - b\| = 0$ which implies that $a = b$.*
From now on, we write $\lim x_n$ for the limit of (x_n) provided the limit exists.

(ii) *The definition of a convergent sequence (x_n) depends on the underling space where the sequence (x_n) sits in. For example, for each $n = 1, 2...$, let $x_n(i) := 1/i$ as $1 \le i \le n$ and $x_n(i) = 0$ as $i > n$. Then (x_n) is a convergent sequence in ℓ^∞ but it is not convergent in c_{00}.*

Definition 1.11. *Let A be a subset of X.*

(i) A point $z \in X$ is called a limit point of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < \|z - a\| < \varepsilon$, that is, $B^*(z, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.
Furthermore, if A contains the set of all its limit points, then A is said to be closed in X.

(ii) The closure of A, write $\overline{A}$, is defined by

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

Remark 1.12. *With the notation as above, it is clear that a point $z \in \overline{A}$ if and only if $B(z, r) \cap A \neq \emptyset$ for all $r > 0$. This is also equivalent to saying that there is a sequence (x_n) in A such that $x_n \to a$. In fact, this can be shown by considering $r = \frac{1}{n}$ for $n = 1, 2....$*

Proposition 1.13. *With the notation as before, we have the following assertions.*

(i) A is closed in X if and only if its complement $X \setminus A$ is open in X.

(ii) The closure $\overline{A}$ is the smallest closed subset of X containing A. The "smallest" in here means that if F is a closed subset containing A, then $\overline{A} \subseteq F$.
Consequently, A is closed if and only if $\overline{A} = A$.

Proof. If A is empty, then the assertions (i) and (ii) both are obvious. Now assume that $A \neq \emptyset$. For part (i), let $C = X \setminus A$ and $b \in C$. Suppose that A is closed in X. If there exists an element $b \in C \setminus int(C)$, then $B(b, r) \nsubseteq C$ for all $r > 0$. This implies that $B(b, r) \cap A \neq \emptyset$ for all $r > 0$ and hence, b is a limit point of A since $b \notin A$. It contradicts to the closeness of A. So, $A = int(A)$ and thus, A is open.

For the converse of (i), assume that C is open in X. Assume that A has a limit point z but $z \notin A$. Since $z \notin A$, $z \in C = int(C)$ because C is open. Hence, we can find $r > 0$ such that $B(z, r) \subseteq C$. This gives $B(z, r) \cap A = \emptyset$. This contradicts to the assumption of z being a limit point of A. So, A must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that $\overline{A}$ is closed. Let z be a limit point of $\overline{A}$. Let $r > 0$. Then there is $w \in B^*(z, r) \cap \overline{A}$. Choose $0 < r_1 < r$ small enough such that $B(w, r_1) \subseteq B^*(z, r)$. Since w is a limit point of A, we have $\emptyset \neq B^*(w, r_1) \cap A \subseteq B^*(z, r) \cap A$. So, z is a limit point of A. Thus, $z \in \overline{A}$ as required. This implies that $\overline{A}$ is closed.

It is clear that $\overline{A}$ is the smallest closed set containing A.

The last assertion follows from the minimality of the closed sets containing A immediately.

The proof is finished. □

Example 1.14. *Retains all notation as above. We have $\overline{c_{00}} = c_0 \subseteq \ell^\infty$.*
Consequently, c_0 is a closed subspace of ℓ^∞ but c_{00} is not.

Proof. We first claim that $\overline{c_{00}} \subseteq c_0$. Let $z \in \ell^\infty$. It suffices to show that if $z \in \overline{c_{00}}$, then $z \in c_0$, that is, $\lim\limits_{i \to \infty} z(i) = 0$. Let $\varepsilon > 0$. Then there is $x \in B(z, \varepsilon) \cap c_{00}$ and hence, we have $|x(i) - z(i)| < \varepsilon$ for all $i = 1, 2.....$ Since $x \in c_{00}$, there is $i_0 \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq i_0$. Therefore, we have $|z(i)| = |z(i) - x(i)| < \varepsilon$ for all $i \geq i_0$. So, $z \in c_0$ as desired.

For the reverse inclusion, let $w \in c_0$. It needs to show that $B(w, r) \cap c_{00} \neq \emptyset$ for all $r > 0$. Let $r > 0$. Since $w \in c_0$, there is i_0 such that $|w(i)| < r$ for all $i \geq i_0$. If we let $x(i) = w(i)$ for $1 \leq i < i_0$ and $x(i) = 0$ for $i \geq i_0$, then $x \in c_{00}$ and $\|x - w\|_\infty := \sup\limits_{i=1,2...} |x(i) - w(i)| < r$ as required. □

2. Lecture 2: Banach Spaces

A sequence (x_n) in X is called a ***Cauchy sequence*** if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. We have the following simple observation.

Lemma 2.1. *Every convergent sequence in X is a Cauchy sequence.*

The following notation plays an important role in mathematics.

Definition 2.2. *A subset A of X is said to be complete if if every Cauchy sequence in A is convergent.*
X is called a **Banach space** *if X is a complete normed space.*

Example 2.3. *With the notation as above, we have the following examples of Banach spaces.*

(i) If $\mathbb{K}^n$ is equipped with the usual norm, then $\mathbb{K}^n$ is a Banach space.

(ii) ℓ^∞ is a Banach space. In fact, if (x_n) is a Cauchy sequence in ℓ^∞, then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, we have

$$|x_n(i) - x_m(i)| \le \|x_n - x_m\|_\infty < \varepsilon$$

*for all $m, n \ge N$ and $i = 1, 2.....$ Thus, if we fix $i = 1, 2, ..,$ then $(x_n(i))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, the limit $\lim_n x_n(i)$ exists in $\mathbb{K}$ for all $i = 1, 2....$ Nor for each $i = 1, 2...$, we put $z(i) := \lim_n x_n(i) \in \mathbb{K}$. Then we have $z \in \ell^\infty$ and $\|z - x_n\|_\infty \to 0$. So, $\lim_n x_n = z \in \ell^\infty$ (**Check !!!!**). Thus ℓ^∞ is a Banach space.*

(iii) ℓ^p is a Banach space for $1 \le p < \infty$. The proof is similar to the case of ℓ^∞.

(iv) $C[a, b]$ is a Banach space.

(v) Let $C_0(\mathbb{R})$ be the space of all continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$ which are vanish at infinity, that is, for every $\varepsilon > 0$, there is a $M > 0$ such that $|f(x)| < \varepsilon$ for all $|x| > M$. Now $C_0(\mathbb{R})$ is endowed with the sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

for every $f \in C_0(\mathbb{R})$. Then $C_0(\mathbb{R})$ is a Banach space.

Proposition 2.4. *Let Y be a subspace of a Banach space X. Then Y is a Banach space if and only if Y is closed in X.*

Proof. For the necessary condition, we assume that Y is a Banach space. Let $z \in \overline{Y}$. Then there is a convergent sequence (y_n) in Y such that $y_n \to z$. Since (y_n) is convergent, it is also a Cauchy sequence in Y. Then (y_n) is also a convergent sequence in Y because Y is a Banach space. So, $z \in Y$. This implies that $\overline{Y} = Y$ and hence, Y is closed.
For the converse statement, assume that Y is closed. Let (z_n) be a Cauchy sequence in Y. Then it is also a Cauchy sequence in X. Since X is complete, $z := \lim z_n$ exists in X. Note that $z \in Y$ because Y is closed. So, (z_n) is convergent in Y. Thus, Y is complete as desired. □

Corollary 2.5. *c_0 is a Banach space but the finite sequence c_{00} is not.*

Proposition 2.6. *Let $(X, \|\cdot\|)$ be a normed space. Then there is a normed space $(X_0, \|\cdot\|_0)$, together with a linear map $i : X \to X_0$, satisfy the following condition.*

(i) X_0 is a Banach space.

(ii) The map i is an isometry, that is, $\|i(x)\|_0 = \|x\|$ for all $x \in X$.

(iii) the image $i(X)$ is dense in X_0, that is, $\overline{i(X)} = X_0$.

Moreover, such pair (X_0, i) is unique up to isometric isomorphism in the following sense: if $(W, \|\cdot\|_1)$ is a Banach space and an isometry $j : X \to W$ is an isometry such that $\overline{j(X)} = W$, then there is an isometric isomorphism ψ from X_0 onto W such that

$$j = \psi \circ i : X \to X_0 \to W.$$

In this case, the pair (X_0, i) is called the completion of X.

Example 2.7. *Proposition 2.6 cannot give an explicit form of the completion of a given normed space. The following examples are basically due to the uniqueness of the completion.*

(*i*) *If X is a Banach space, then the completion of X is itself.*

(*ii*) *By Corollary 2.5, the completion of the finite sequence space c_{00} is the null sequence space c_0.*

(*iii*) *The completion of $C_c(\mathbb{R})$ is $C_0(\mathbb{R})$.*

Definition 2.8. *A subset A of a normed space X is said to be nowhere dense in X if $int(\overline{A}) = \emptyset$.*

Example 2.9.

(*i*) *The set of all integers $\mathbb{Z}$ is a nowhere dense subset of $\mathbb{R}$.*

(*ii*) *The set $(0,1)$ is a nowhere dense subset of $\mathbb{R}^2$ but it is not a nowhere dense subset of $\mathbb{R}$.*

(*iii*) *Let $A := \{x \in c_{00} : x(n) \geq 0,\ for\ all\ n = 1,2...\}$. Notice that A is a closed subset of c_{00}. We claim that $int(A) = \emptyset$. In fact, let $a \in A$ and $r > 0$. Since $a \in c_{00}$, there is N such that $a(n) = 0$ for all $n \geq N$. Now define $z \in c_{00}$ by $z(n) = x(n)$ for $n \neq N$ and $z(N) := \frac{-r}{2}$. Then $z \in c_{00} \setminus A$ and $\|z - a\|_\infty < r$. So, $int(A) = \emptyset$ and thus, A is a nowhere dense subset of c_{00}.*

Lemma 2.10. *Let X be a Banach space. We have the following assertions.*

(*i*) *A subset A of X is nowhere dense in X if and only if the complement of $\overline{A}$ is an open dense subset of X.*

(*ii*) *If (W_n) is a sequence of open dense subsets of X, then $\bigcap_{n=1}^{\infty} W_n \neq \emptyset$.*

Proof. For (*i*), let $z \in X$ and $r > 0$. It is clear that we have $B(z,r) \nsubseteq \overline{A}$ if and For (*ii*), we first fix an element $x_1 \in W_1$. Since W_1 is open, then there is $r_1 > 0$ such that $B(x_1, r_1) \subseteq W_1$. Notice that since W_2 is open dense in X, we can find an element $x_2 \in B(x_1, r_1) \cap W_2$ and $0 < r_2 < r_1/2$ such that $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap W_2$. To repeat the same step, we can get a sequence of element (x_n) in X and a sequence of positive numbers (r_n) such that

(*a*) $r_{k+1} < r_k/2$, and

(*b*) $\overline{B(x_{k+1}, r_{k+1})} \subseteq B(x_k, r_k) \cap W_{k+1}$

for all $k = 1, 2, ...$

From this, we see that (x_k) is a Cauchy sequence in X. Then by the completeness of X, $\lim x_k = a$ exists in X. It remains to show that $a \in \bigcap W_k$. Fix N. Note that by the condition (*b*) above, we see that $x_k \in \overline{B(x_N, r_N)} \subseteq B(x_{N-1}, r_{N-1}) \cap W_N$ for all $k > N$. Since $\overline{B(x_N, r_N)}$ is closed, we see that $a = \lim x_k \in \overline{B(x_N, r_N)}$. This implies that $a \in W_N$. Therefore, $\bigcap W_k$ is non-empty as required. ☐

Theorem 2.11. Baire Category Theorem*: Let X be a Banach space. Suppose that $X = \bigcup_{n=1}^{\infty} A_n$ for a sequence of subsets (A_n) of X. Then there is A_{n_0} not nowhere dense in X.*

Proof. Suppose that each A_n is nowhere dense in X. If we put $W_n := \overline{A}_n^c$, then each W_n is an open dense subset of X by Lemma 2.10 (*i*). Lemma 2.10 (ii) implies that $\bigcap W_n \neq \emptyset$. This gives

$$X \supsetneq \left(\bigcap W_n\right)^c = \bigcup W_n^c = \bigcup \overline{A}_n \supseteq \bigcup A_n = X.$$

This leads to a contradiction. The proof is finished. ☐

3. Lecture 3: Series in normed spaces

Throughout this section, let X be a normed space.

Let (x_n) be a sequence elements in X. Now for each $n = 1, 2, ..$, put $s_n = x_1 + \cdots + x_n$ and call the n-th *partial sum* of a formal series $\sum_{n=1}^{\infty} x_n$.

Definition 3.1. *With the notation as above, we say that a series* $\sum_{n=1}^{\infty} x_n$ *is convergent in* X *if the sequence of the sequence of partial sums* (s_n) *is convergent in* X*. In this case, we also write*

$$\sum_{n=1}^{\infty} x_n := \lim_n s_n \in X.$$

Moreover, we say that a series $\sum_{n=1}^{\infty} x_n$ *is absolutely convergent in* X *if* $\sum_{n=1}^{\infty} \|x_n\| < \infty$*.*

Lemma 3.2. *Let* (x_n) *be a Cauchy sequence in a normed space* X*. If* (x_n) *has a convergent subsequence in* X*, then* (x_n) *itself is convergent too.*

Proof. Let (x_{n_k}) be a convergent subsequence of (x_n) and let $L := \lim_k x_{n_k} \in X$. We are going to show that $\lim_n x_n = L$.
Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. On the other hand, since $\lim_k x_{n_k} = L$, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and $\|L - x_{n_K}\| < \varepsilon$. Thus, if $n \geq n_K$, we see that $\|x_n - L\| \leq \|x_n - x_{n_K}\| + \|x_{n_K} - L\| < 2\varepsilon$. The proof is finished. □

Proposition 3.3. *Let* X *be a normed space. Then the following statements are equivalent.*

(i) X *is a Banach space.*
(ii) Every absolutely convergent series in X *is convergent.*

Proof. For showing $(i) \Rightarrow (ii)$, assume that X is a Banach space and let $\sum x_k$ be an absolutely convergent series in X. Put $s_n := \sum_{k=1}^{n} x_k$ the n-th partial sum of $\sum x_k$. Let $\varepsilon > 0$. Since the series $\sum_k x_k$ is absolutely convergent, there is $N \in \mathbb{N}$ such that $\sum\limits_{n+1 \leq k \leq n+p} \|x_k\| < \varepsilon$ for all $n \geq N$ and $p = 1, 2.....$ This gives $\|s_{n+p} - s_n\| \leq \sum\limits_{n+1 \leq k \leq n+p} \|x_k\| < \varepsilon$ for all $n \geq N$ and $p = 1, 2.....$ Thus, (s_n) is a Cauchy sequence in X. Then by the completeness of X, we see that the series $\sum x_k$ is convergent in X as desired.
Now suppose that the condition (ii) holds. Let (x_n) be a Cauchy sequence in X. Notice that by the definition of a Cauchy sequence, we can find a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_{k+1}} - x_{n_k}\| < 1/2^k$ for all $k = 1, 2......$ From this, we see that the series $\sum_k (x_{n_{k+1}} - x_{n_k})$ is absolutely convergent in X. Then the condition (ii) tells us that the series $\sum_k (x_{n_{k+1}} - x_{n_k})$ is convergent in X. Notice that $x_{n_m} = x_{n_1} + \sum\limits_{k=1}^{m} (x_{n_{k+1}} - x_{n_k})$ for all $m = 1, 2, ...$ Therefore, $(x_{n_k})_{k=1}^{\infty}$ is a convergent subsequence of (x_n). Then by Lemma 3.2, we see that (x_n) is convergent in X. The proof is finished.

□

Recall that a *basis* of a vector space V over $\mathbb{K}$ is a collection of vectors in V, say $(v_i)_{i \in I}$, such that for each element $x \in V$, we have a unique expression

$$x = \sum_{i \in I} \alpha_i v_i$$

for some $\alpha_i \in \mathbb{K}$ and **all** $\alpha_i = 0$ **except finitely many** α_i**'s**.
One of fundamental properties of a vector space is that **every vector space must have a basis.** The proof of this assertion is due to the *Zorn's lemma*.

Definition 3.4. *A sequence* (x_n) *is called a Schauder basis for a normed space* X *if for each element* $x \in X$*, there is a unique sequence* (α_n) *in* $\mathbb{K}$ *such that*

$$x = \sum_{n=1}^{\infty} \alpha_n x_n. \tag{3.1}$$

Remark 3.5.

(i) Notice that a Schauder basis must be linearly independent vectors. So, it is clear that every Schauder basis is a vector basis for a finite dimensional vector space. However, a Schauder basis need not be a vector basis for a normed space in general. For example, if we consider the sequence (e_n) in c_0 given by $e_n(n) = 1$; otherwise, $e_n(i) = 0$, then (e_n) is a Schauder basis for c_0 but it it is not a vector basis.

(ii) In the Definition 3.4, the expression 3.1 depends on the order of (x_n). More precise, if we are given a bijection $\sigma : \{1, 2...\} \to \{1, 2...\}$, then the Eq 3.1 CANNOT assure that we still have the expression $x = \sum_{n=1}^{\infty} \alpha_{\sigma(n)} x_{\sigma(n)}$ for each $x \in X$.

Example 3.6. *(i) If* X *is of finite dimension, then the vector bases are the same as the Schauder bases.*

(ii) Let e_n *be a sequence defined as in Remark 3.5(i), then the sequence* (e_n) *is a Schauder basis for the spaces* c_0 *and* ℓ^p *for* $1 \le p < \infty$*.*

Definition 3.7. *A normed space* X *is said to be separable if there is a countable dense subset of* X*.*

Example 3.8. *(i) The space* $\mathbb{C}^n$ *is separable. In fact, it is clear that* $(\mathbb{Q} + i\mathbb{Q})^n$ *is a countable dense subset of* $\mathbb{C}^n$*.*

(ii) The space ℓ^∞ *is an important example of nonseparable Banach space. In fact, if we put* $D := \{x \in \ell^\infty : x(i) = 0 \text{ or } 1\}$*, then* D *is an uncountable subset of* ℓ^∞*. Moreover, we have* $\|x - y\|_\infty = 1$ *for any* $x, y \in D$ *with* $x \ne y$*. Thus,* $\{B(x, 1/2) : x \in D\}$ *is an uncountable family of disjoint open balls of* ℓ^∞*. So, if* C *is a countable dense subset of* ℓ^∞*, then* $C \cap B(x, 1/2) \ne \emptyset$ *for all* $x \in D$*. Also, for each element* $z \in C$*, there is a unique element* $x \in D$ *such that* $z \in B(x, 1/2)$*. It leads to a contradiction since* D *is uncountable. Therefore,* ℓ^∞ *is nonseparable.*

Proposition 3.9. *Let* X *be a normed space. Then* X *is separable if and only if there is a countable subset* A *of* X *such that the linear span of* A *is dense in* X*, that is, for any element* $x \in X$ *and* $\varepsilon > 0$*, there are finite many elements* $x_1, .., x_N$ *in* A *such that* $\|x - \sum_{k=1}^{N} \alpha_k x_k\| < \varepsilon$ *for some scalars* $\alpha_1, .., \alpha_N$*.*
Consequently, if X *has a Schauder basis, then* X *is separable.*

Proof. The necessary condition is clear.
We are now going to prove the converse statement. Suppose that X is the closed linear span of a countable subset A. Now let D be the linear span of A over the field $\mathbb{Q} + i\mathbb{Q}$. Since $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, this implies that D is a countable dense subset of X. Thus, X is separable.
The last statement is clearly follows from the definition of a Schauder basis at once. □

By Proposition 3.9, we have the following important examples of separable Banach spaces at once.

Corollary 3.10. *The spaces* c_0 *and* ℓ^p *for* $1 \le p < \infty$ *all are separable.*

Remark 3.11. Proposition 3.9 leads to the following natural question which was first raised by Banach (1932).
The Basis Problem: Does every separable Banach space have a Schauder basis?
The answer is ″**No**″.
This problem was completely solved by P. Enflo in 1973.

References

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